Quivers, desingularizations and canonical bases

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Abstract

A class of desingularizations for orbit closures of representations of Dynkin quivers is constructed, which can be viewed as a graded analogue of the Springer resolution. A stratification of the singular fibres is introduced; its geometry and combinatorics are studied. Via the Hall algebra approach, these constructions relate to bases of quantized enveloping algebras. Using Ginzburg's theory of convolution algebras, the base change coefficients of Lusztig's canonical basis are expressed as decomposition numbers of certain convolution algebras.

1 Introduction

The varieties of representations of Dynkin quivers are of central importance to geometric realizations of quantized enveloping algebras and structures related to them. For example, positive parts of quantized enveloping algebras can be realized by a convolution construction (C. M. Ringel's Hall algebra approach [Ri2]). Of particular interest are orbit closures in these varieties, since their intersection cohomology realizes G. Lusztig's canonical bases ([Lu1]). On the other hand, these varieties can be viewed as (quiver-) graded analogues of the nilpotent cones, which makes their geometric analysis interesting in itself.

Motivated by this last analogy, this paper starts a program to develop an analogue of Springer theory of nilpotent cones in the quiver context, and to explore the quantum group theoretic consequences of such constructions.

As a first step, a model for Springer's resolution has to be found. Therefore, we construct in section 2 desingularizations of orbit closures for representations of arbitrary Dynkin quivers, generalizing the case of equioriented quivers of type A treated in [ADK]. At the heart of this construction lies the Auslander-Reiten theory of finite dimensional algebras.

When a desingularization of a variety is known, one can study its geometry

by studying the singular fibres. In our case, we introduce a stratification in section 3, which can be viewed as an analogue of Spaltenstein's stratification ([Sp]). Its geometric and combinatorial aspects are studied in section 4, where a combinatorial approach to quiver representations and crystal basis, developed in [Re3], plays a key role. As a consequence, certain important differences to the nilpotent cone case are established.

Via the Hall algebra approach, the meaning of our geometric constructions for quantum groups is discussed in section 5. The main result is a geometric interpretation of the monomial bases introduced in [Re1].

In section 6, we develop a first instance of Springer theory in the quiver setup: V. Ginzburg's theory of representations of convolution algebras ([CG]) is applied to a quiver analogue of the Steinberg triple variety. As a result, we get a 'Kazhdan-Lusztig type' statement, relating certain base change coefficients in quantum groups to decomposition numbers for representations of certain convolution algebras. The nature of these algebras is, however, unknown at the moment, and will be explored in future work.

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2 Construction of desingularizations

The basic setup for the construction of desingularizations is a study of the relation between varieties of quiver representations and of flags in graded vector spaces. This setup is also central in Lusztig's construction of quantum groups using perverse sheaves (see [Lu3]); we will review it here, thereby fixing the notations. For general facts concerning representations of quivers and their geometry, the reader is referred to [Bo], [Ri1].

Let Q be a finite Dynkin quiver, i.e. an oriented graph with finite sets of vertices I and arrows Q_1, whose underlying unoriented graph is a disjoint union of Dynkin diagrams of type A, D, E. Let k be a field. For a finite dimensional I-graded k-vector space $V = \oplus_{i \in I} V_i$, we call the formal sum $d = \underline{\dim}\, V = \sum_i (\dim_k V_i) i \in \mathbf{N}I$ the dimension vector of V; thus, $V \simeq k^d = \oplus_i k^{d_i}$, where d_i denotes the i-th component of d. The vector space V is called pure of weight i if $V_j = 0$ for all $j \neq i$.
A pair $(\mathbf{i}, \mathbf{a})$ of finite sequences $\mathbf{i} = (i_1, \ldots, i_\nu) \in I^\nu$, $\mathbf{a} = (a_1, \ldots, a_\nu) \in \mathbf{N}^\nu$ is called a monomial in I; its weight is defined as $d = |(\mathbf{i}, \mathbf{a})| = \sum_k a_k i_k \in \mathbf{N}I$. A

flag F^* of I-graded subspaces

$$k^d = F^0 \supset F^1 \supset \ldots \supset F^\nu = 0$$

is called of type $(\mathbf{i}, \mathbf{a})$ if F^{k-1}/F^k is pure of weight i_k and dimension a_k, for all $k = 1 \ldots \nu$. The set $\mathcal{F}_{\mathbf{i},\mathbf{a}}$ of all flags of type $(\mathbf{i}, \mathbf{a})$ is acted upon transitively by the group $G_d = \prod_{i \in I} \mathrm{GL}(k^{d_i}) \subset \mathrm{GL}(k^d)$. We fix once and for all an arbitrary flag $F_0^* \subset \mathcal{F}_{\mathbf{i},\mathbf{a}}$ and denote by $P_{\mathbf{i},\mathbf{a}}$ its stabilizer under the G_d-action, a parabolic subgroup of dimension $\sum_{k \leq l\, : i_k = i_l} a_k a_l$. Thus, $\mathcal{F}_{\mathbf{i},\mathbf{a}} \simeq G_d/P_{\mathbf{i},\mathbf{a}}$ is a projective algebraic variety of dimension $\sum_{k < l\, : i_k = i_l} a_k a_l$.

We denote by R_d the variety

$$R_d = \bigoplus_{\alpha : i \to j} \mathrm{Hom}_k(k^{d_i}, k^{d_j}) \subset \mathrm{End}_k(k^d).$$

This affine variety can be viewed on the one hand as the parameter space for k-representations of the quiver Q of dimension vector d, on the other hand as the variety of 'Q_1-graded' nilpotent endomorphisms of k^d (the nilpotency follows since Q has no oriented cycles). The action of $\mathrm{GL}_k(k^d)$ on $\mathrm{End}_k(k^d)$ by conjugation restricts to an action of G_d on R_d via

$$(g_i)_i (M_\alpha)_\alpha (g_i^{-1})_i = (g_j M_\alpha g_i^{-1})_{\alpha : i \to j}.$$

The orbits $\mathcal{O}_M$ for this action are in one-to-one correspondence with the isomorphism classes $[M]$ of quiver representations of dimension vector d. Denote by $\mathrm{mod} kQ$ the category of finite dimensional k-representations of Q. By Gabriel's theorem, the isomorphism classes of indecomposable objects X_α in $\mathrm{mod}\, kQ$ are in bijection with the set R^+ of positive roots for the Dynkin diagram corresponding to Q via $\underline{\dim}\, X_\alpha = \alpha$, where R^+ is identified with a subset of $\mathbf{N}I$ by identifying the simple root to the vertex $i \in I$ with $i \in \mathbf{N}I$. Thus, the set $[\mathrm{mod} kQ]$ of isoclasses in $\mathrm{mod} kQ$ is in bijection with the set of functions $R^+ \to \mathbf{N}$. Consequently, the number of G_d-orbits in R_d is finite for all $d \in \mathbf{N}I$.

Define $X_{\mathbf{i},\mathbf{a}}$ as the set of pairs $(M, F^*) \in R_d \times \mathcal{F}_{\mathbf{i},\mathbf{a}}$ such that M and F^* are compatible, i.e. $M(F^k) \subset F^k$ for all $k = 0 \ldots \nu$, where M is viewed as an endomorphism of k^d. Let $Y_{\mathbf{i},\mathbf{a}}$ be the subset of R_d consisting of those M which are compatible with the fixed flag F_0^*. It is easy to see that $Y_{\mathbf{i},\mathbf{a}}$ is a k-subspace of R_d of dimension $\sum_{k < l\, : i_k \to i_l} a_k a_l$. Using this notation, we can identify $X_{\mathbf{i},\mathbf{a}}$ with the associated fibre bundle $G_d \times^{P_{\mathbf{i},\mathbf{a}}} Y_{\mathbf{i},\mathbf{a}}$ via the map

$$\overline{(g, M)} \mapsto (gMg^{-1}, gF_0^*).$$

From this observation we get immediately the following:

1. $X_{\mathbf{i},\mathbf{a}}$ is an irreducible smooth algebraic variety of dimension

$$\dim X_{\mathbf{i},\mathbf{a}} = \sum_{\substack{k<l \\ i_k = i_l}} a_k a_l + \sum_{\substack{k<l \\ i_k \to i_l}} a_k a_l,$$

2. the map $X_{\mathbf{i},\mathbf{a}} \rightarrow \mathcal{F}_{\mathbf{i},\mathbf{a}}$ is a homogeneous vector bundle with typical fibre $Y_{\mathbf{i},\mathbf{a}}$,

3. the canonical map $\pi_{\mathbf{i},\mathbf{a}} : X_{\mathbf{i},\mathbf{a}} \rightarrow R_d$ is a projective morphism (a collapsing of a homogeneous bundle in the sense of G. Kempf).

By the definitions, the image of $\pi_{\mathbf{i},\mathbf{a}}$ consists of those representations M of Q which possess a chain of subrepresentations

$$M = M^0 \supset M^1 \supset \ldots \supset M^\nu = 0$$

such that for all $k = 1 \ldots \nu$, the subquotient M^{k-1}/M^k is isomorphic to $E_{i_k}^{a_k}$, the a_k-fold direct sum of the simple object $E_{i_k} \in \operatorname{mod} kQ$ (since this is the only representation of dimension vector $a_k i_k$).

Since $\pi_{\mathbf{i},\mathbf{a}}$ is projective, its image is closed. It is irreducible since $X_{\mathbf{i},\mathbf{a}}$ is, and obviously G_d-stable; thus $\pi_{\mathbf{i},\mathbf{a}}(X_{\mathbf{i},\mathbf{a}})$ is one orbit closure $\overline{\mathcal{O}_M}$. The fibre of $\pi_{\mathbf{i},\mathbf{a}}$ over M identifies with the set of flags of type $(\mathbf{i},\mathbf{a})$ which are compatible with M. These facts suggest the following strategy for producing a desingularization of $\overline{\mathcal{O}_M}$: we need to find a monomial $(\mathbf{i},\mathbf{a})$ such that M possesses a unique filtration (as above) of type $(\mathbf{i},\mathbf{a})$, and is the generic representation with this property.

To do this, we translate the construction of monomial bases for quantized enveloping algebras from [Re1] into the present geometric setup; the relation between the algebraic approach of [Re1] and the present geometric approach is explained in section 5. For the reader's convenience, we recall the main steps of this construction. We start by recalling the concept of a directed partition of R^+.

Definition 2.1 *A partition $\mathcal{I}_* = (\mathcal{I}_1, \ldots, \mathcal{I}_s)$, where $R^+ = \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_s$ is called directed if*

1. $\operatorname{Ext}^1(X_\alpha, X_\beta) = 0$ *for all* $\alpha, \beta \in \mathcal{I}_t$ *for* $t = 1 \ldots s$,

2. $\operatorname{Hom}(X_\beta, X_\alpha) = 0 = \operatorname{Ext}^1(X_\alpha, X_\beta)$ *for all* $\alpha \in \mathcal{I}_t$, $\beta \in \mathcal{I}_u$, $t < u$.

Remarks:

1. Using the fact that the category modkQ is representation-directed (see [Ri1]), we see that directed partitions do always exist.

2. The above conditions can also be written in purely root-theoretic terms using the non-symmetric bilinear form on $\mathbf{N}I$ defined by

$$\langle i,j \rangle = \begin{cases} 1 & , \quad i = j, \\ -1 & , \quad i \rightarrow j \text{ in } Q, \\ 0 & , \quad \text{otherwise,} \end{cases}$$

which fulfills

$$\langle \underline{\dim}\, M, \underline{\dim}\, N\rangle = \dim \mathrm{Hom}(M,N) - \dim \mathrm{Ext}^1(M,N).$$

Namely, the definition is equivalent to:

(a) $\langle \alpha, \beta\rangle \geq 0$ for all $\alpha, \beta \in \mathcal{I}_t$, $t = 1 \ldots s$,

(b) $\langle \alpha, \beta\rangle \geq 0 \geq \langle \beta, \alpha\rangle$ for all $\alpha \in \mathcal{I}_t$, $\beta \in \mathcal{I}_u$, $t < u$.

This can be derived from the following facts (see [Ri1]): there exists a partial ordering $\preceq$ on the indecomposable representations of Q such that the non-vanishing of $\mathrm{Hom}(U,V)$ already implies $U \preceq V$, and we have $\dim \mathrm{Ext}^1(U,V) = \dim \mathrm{Hom}(V, \tau U)$ by the Auslander-Reiten formula, where τ denotes the Auslander-Reiten translation on modkQ.

We will now construct a pair consisting of a sequence $\mathbf{i}$ and an additive function $\mathbf{a} : [\mathrm{mod}kQ] \to \mathbf{N}^\nu$ from a directed partition $\mathcal{I}_* = (\mathcal{I}_1, \ldots, \mathcal{I}_s)$, which will be fixed from now on. This pair will be called the monomial function associated to $\mathcal{I}_*$. We choose a total ordering on I such that the existence of an arrow $i \to j$ implies $i < j$ (which is possible since there are no oriented cycles in Q). For $t = 1 \ldots s$, we define a sequence ω_t in I by writing the elements of the set

$$\{i \in I\,:\, \alpha_i \neq 0 \text{ for some } \alpha \in \mathcal{I}_t\}$$

in ascending order with respect to the chosen ordering on I. We define the sequence $\mathbf{i}$ as the concatenation $\mathbf{i} = \omega_1 \ldots \omega_s$. To construct the function $\mathbf{a} : [\mathrm{mod}kQ] \to \mathbf{N}^\nu$ on a given representation M, we define $M_{(t)}$ as the direct sum of all direct summands U of M which are isomorphic to some X_α for $\alpha \in \mathcal{I}_t$, for $t = 1 \ldots s$. Thus, we arrive at a decomposition

$$M = M_{(1)} \oplus \ldots \oplus M_{(s)}.$$

For each $t = 1 \ldots s$, we write the word ω_t as $(i_1 \ldots i_u)$ and define the sequence $\mathbf{a}_t(M)$ as

$$(\underline{\dim}_{\,i_1} M_{(t)}, \ldots, \dim_{i_u} M_{(t)}).$$

The sequence $\mathbf{a}(M)$ is given by concatenation: $\mathbf{a}(M) = (\mathbf{a}_1(M), \ldots, \mathbf{a}_s(M))$. It is immediately clear that the function $\mathbf{a}$ is additive, i.e. $\mathbf{a}(M \oplus N) = \mathbf{a}(M) + \mathbf{a}(N)$ (componentwise addition). We can now formulate the main theorem of this section.

Theorem 2.2 *For each representation M, the map*

$$\pi_M = \pi_{\mathbf{i},\mathbf{a}(M)} : X_M = X_{\mathbf{i},\mathbf{a}(M)} \to R_d$$

has the orbit closure $\overline{\mathcal{O}_M}$ as its image, and it restricts to an isomorphism $\pi_M : \pi_M^{-1}(\mathcal{O}_M) \simeq \mathcal{O}_M$ over the orbit $\mathcal{O}_M$. Thus, π_M is a desingularization of the orbit closure $\overline{\mathcal{O}_M}$.

Proof: For the calculation of the image of π_M, we use Lemma 4.4 of [Re1] (see also Proposition 2.4 of [Re2]), which states in particular the following:

Let M and N be representations of Q such that $\mathrm{Ext}^1(M,N)=0$. Let M' and N' be degenerations of M and N, respectively, and let X be an extension of M' by N'. Then X is a degeneration of $M\oplus N$. If, moreover, $\mathrm{Hom}(N,M)=0$ and $X\simeq M\oplus N$, then $M'\simeq M$ and $N'\simeq N$.

Here, a representation M is said to degenerate to N if the point $N\in R_d$ belongs to the orbit closure $\overline{\mathcal{O}_M}$; in this case we write $M\leq N$. By repeated application, this statement immediately generalizes to the following one:

Lemma 2.3 *If $M_{(1)},\ldots,M_{(s)}\in \mathrm{mod}kQ$ satisfy $\mathrm{Ext}^1(M_{(k)},M_{(l)})=0$ for all $k<l$, and if X is a representation possessing a filtration*

$$X=X^0\supset X^1\supset\ldots\supset X^s=0$$

such that X^{t-1}/X^t is a degeneration of $M_{(t)}$ for $t=1\ldots s$, then X is a degeneration of the direct sum $M_{(1)}\oplus\ldots\oplus M_{(s)}$. If, moreover, $\mathrm{Hom}(M_{(k)},M_{(l)})=0$ for $k>l$ and $X\simeq M_{(1)}\oplus\ldots\oplus M_{(s)}$, then $X^{t-1}/X^t\simeq M_{(t)}$ for all $t=1\ldots s$.

Now suppose a representation X belongs to the image of π_M. By definition, X has a filtration

$$X=X_0\supset X_1\supset\ldots\supset X_s=0$$

such that

$$X_{k-1}/X_k\simeq E_{i_k}^{\mathbf{a}(M)_k}\text{ for }k=1\ldots\nu.$$

By writing $\mathbf{i}$ and $\mathbf{a}(M)$ in blocks $\mathbf{i}=\omega_1\ldots\omega_s$, $\mathbf{a}(M)=(\mathbf{a}_1(M),\ldots,\mathbf{a}_s(M))$ as in their above definition, we can coarsen this filtration to another one

$$X=X^0\supset X^1\supset\ldots\supset X^s=0$$

such that X^{t-1}/X^t is of dimension vector

$$\sum_{v=1}^{u}\underline{\dim}_{\,i_v}M_{(t)}=\underline{\dim}\,M_{(t)}\text{ for }t=1\ldots s,$$

where $\omega_t=(i_1\ldots i_u)$. Since $\mathrm{Ext}^1(X_\alpha,X_\beta)=0$ for all $\alpha,\beta\in\mathcal{I}_t$ by the definition of a directed partition, we have $\mathrm{Ext}^1(M_{(t)},M_{(t)})=0$, which means that the orbit of $M_{(t)}$ is dense in its variety of representations. Thus, each representation of dimension vector $\underline{\dim}\,M_{(t)}$ is a degeneration of $M_{(t)}$, which applies in particular to the X^{t-1}/X^t. Using again the definition of a directed partition, we have $\mathrm{Ext}^1(M_{(t)},M_{(u)})=0$ for all $t<u$. Thus, we can apply Lemma 2.3 to conclude that X is a degeneration of $M_{(1)}\oplus\ldots\oplus M_{(s)}=M$, which proves that the image of π_M is contained on $\overline{\mathcal{O}_M}$. Since we already know that the image of π_M is

an orbit closure, we only need to show that M itself belongs to it. We have a filtration

$$M = M_{(1)} \oplus \ldots \oplus M_{(s)} \supset M_{(2)} \oplus \ldots \oplus M_{(s)} \supset \ldots \supset M_{(s)} \supset 0$$

by definition. On the other hand, it is easy to see that each representation X of Q has a unique filtration

$$X = X_0 \supset X_1 \supset \ldots \supset X_m = 0$$

such that

$$X_{p-1}/X_p \simeq E_{i_p}^{b_p} \text{ for } p = 1 \ldots m,$$

where $i_1 \ldots i_m$ is an enumeration of the vertices $i_p \in I$ such that $b_p = \underline{\dim}_{\,i_p} X \neq 0$, ascending with respect to the chosen ordering on I. In particular, this can be applied to the subfactors $M_{(k)}$ of the filtration above. Refining this filtration in this way, it is clear from the definitions that we arrive at a filtration of M of type $(\mathbf{i}, \mathbf{a}(M))$, proving the first part of the theorem.

To prove the second part, it suffices to show that π_M is bijective over $\mathcal{O}_M$: indeed, the orbit $\mathcal{O}_M$ is isomorphic to the homogeneous space $G_d/\mathrm{Aut}_{kQ}(M)$, and the restriction of π_M to the inverse image of the orbit is thus the projection from an associated fibre bundle to a homogeneous space. So assume we are given a filtration of M of type $(\mathbf{i}, \mathbf{a}(M))$. Coarsening it as above, we obtain a filtration

$$M = M^0 \supset M^1 \supset \ldots \supset M^s = 0$$

such that M^{t-1}/M^t is of dimension vector $\underline{\dim}\, M_{(t)}$, i.e. a degeneration of $M_{(t)}$, for $t = 1 \ldots s$. Since $\mathrm{Hom}(M_{u)}, M_{(t)}) = 0$ for $t < u$ by the properties of a directed partition, we can apply the second part of Lemma 2.3 to conclude that $M^{t-1}/M^t \simeq M_{(t)}$ for $t = 1 \ldots s$. Using the Ext-vanishing property for the $M_{(t)}$, an easy induction yields

$$M^t \simeq M_{(t+1)} \oplus \ldots \oplus M_{(s)} \text{ for } t = 1 \ldots s.$$

But by the Hom-vanishing property, this filtration is unique. As already remarked above, the intermediate steps of the original filtration of M are also unique for general reasons. Thus, the whole filtration is unique, proving the second part of the theorem.

$\square$

Examples: As a first example, we show that our desingularizations include as a special case the desingularization for equioriented quivers of type A from [ADK]. So let Q be the quiver $1 \to 2 \to \ldots \to n$. It is known (compare [Re1], section 8) that

$$\mathcal{I}_t = \{\alpha_{n+1-t} + \ldots + \alpha_k \,:\, k = n+1-t, \ldots, n\}$$

defines a directed partition of $R^+ = \{\alpha_i + \ldots + \alpha_j \, : \, 1 \leq i \leq j \leq n\}$, where α_i denotes the simple root corresponding to the vertex $i = 1 \ldots n$. We denote by m_{ij} the multiplicity of the indecomposable representation $E_{ij} = X_{\alpha_i + \ldots + \alpha_j}$ in a representation M for $1 \leq i \leq j \leq n$. Then the partition $\mathcal{I}_*$ induces the monomial function given by

$$\mathbf{i} = (n,\ n-1, n\ , \ldots,\ i, \ldots, n\ , \ldots,\ 1, \ldots, n)$$

and

$$\mathbf{a}(M) = (m_{nn},\ m_{n-1,n-1} + m_{n-1,n}, m_{n-1,n}\ , \ldots,\ m_{ii} + \ldots + m_{in}, \ldots, m_{in}\ , \ldots$$

$$\ldots,\ m_{11} + \ldots + m_{1n}, \ldots, m_{1n}).$$

A short calculation shows that $\mathcal{F}_M = \mathcal{F}_{\mathbf{i},\mathbf{a}(M)}$ thus consists of tuples of flags

$$(k^{d_1} = F_1^0 \supset F_1^1 = 0, k^{d_2} = F_2^0 \supset F_2^1 \supset F_2^2 = 0, \ldots$$

$$, \ldots k^{d_n} = F_n^0 \supset F_n^1 \supset \ldots \supset F_n^n = 0)$$

such that

$$\dim F_i^t = \sum_{j+t \leq i \leq k} m_{jk}.$$

The variety X_M consists of tuples

$$(C_1, \ldots, C_{n-1}, F_*^*) \text{ such that } C_i F_i^* \subset F_{i+1}^* \text{ for all } i = 1 \ldots n-1,$$

where C_i is a $d_{i+1} \times d_i$-matrix for $i = 1 \ldots n-1$, and F_*^* is a tuple of flags as above. Since the above formula for $\dim F_i^t$ can be interpreted as

$$\dim F_i^t = \mathrm{rank}(C_{i-1} \cdot \ldots \cdot C_{i-t}),$$

this is precisely the definition of [ADK].

Desingularizations for the same quiver from other directed partitions involve compatibility conditions between certain kernels and images of products of the matrices C_i; they give an explicit realization of the 'mixed' desingularizations predicted in [Ze].

As a second example, we sketch a construction of a desingularization for the orbit closures of the diagonal action of GL_N on a triple product of Grassmanians

$$\mathrm{GL}_N : X = \mathrm{Gr}_{d_1}^N \times \mathrm{Gr}_{d_2}^N \times \mathrm{Gr}_{d_3}^N$$

for integers $0 \leq d_1, d_2, d_3 \leq N$; the details will be left to the reader. We consider the quiver Q of type D_4 with set of vertices $I = \{0, 1, 2, 3\}$ and arrows α_i pointing from i to 0 for $i = 1, 2, 3$, respectively. We define a dimension vector $d = N \cdot 0 + d_1 \cdot 1 + d_2 \cdot 2 + d_3 \cdot 3 \in \mathbf{N}I$ and consider the open subvariety R_d^0

of the representation variety R_d consisting of tuples (M_{α_i}) such that all M_{α_i} are injective maps. The subgroup $\mathrm{GL}_{d_1} \times \mathrm{GL}_{d_2} \times \mathrm{GL}_{d_3}$ of G_d acts freely on R_d^0 with geometric quotient X. The G_d-action on R_d induces the diagonal GL_N-action on X, so that we can work with the action of G_d on R_d^0 equally. Using this relation, the orbits can be described as in ([Ka], 1.17.). In particular, the multiplicities of the indecomposables X_α in a representation M corresponding to a triple of subspaces can be described in terms of dimensions of sums and intersections of the U_i. A directed partition of R^+ is given by

$$\mathcal{I}_1 = \{\alpha_0\},\ \mathcal{I}_2 = \{2\alpha_0 + \sum_{i=1}^{3} \alpha_i, \alpha_0 + \alpha_i\ :\ i = 1, 2, 3\},$$

$$\mathcal{I}_3 = \{\alpha_0 + \sum_{i=1}^{3} \alpha_i, \alpha_0 + \sum_{i=1}^{3} \alpha_i - \alpha_j\ :\ j = 1, 2, 3\},\ \mathcal{I}_4 = \{\alpha_i\ :\ i = 1, 2, 3\}.$$

Using this information, one can construct the desingularization of $\overline{\mathcal{O}_M}$ and translate it to the product of Grassmanians, yielding the following result:
A desingularization of the $\mathrm{GL}(N)$-orbit closure of a triple (U_1, U_2, U_3) is given by the projection to the first three factors of the smooth variety

$$\{(U_1, \ldots, U_8) \in \prod_{i=1}^{8} \mathrm{Gr}_{d_i}^N\ :$$

$$U_{i+3} \subset U_i \text{ for } i = 1, 2, 3,\ U_1 + U_2 + U_3 \subset U_7,\ U_4 + U_5 + U_6 \subset U_8,\ U_8 \subset U_7\},$$

where $\dim U_{i+3} = \sum_{j \neq i} \dim U_i \cap U_j - \dim U_1 \cap U_2 \cap U_3$ for $i = 1, 2, 3$, $\dim U_7 = \dim U_1 + U_2 + U_3$, $\dim U_8 = \sum_{j \neq k} \dim U_j \cap U_k - \dim U_1 \cap U_2 \cap U_3$.

Returning to the general case, we briefly discuss the following:
Since there are many possible directed partitions for a given quiver Q, we have constructed a whole class of desingularizations of the orbit closures. The question arises how these different desingularizations are related.

Call a directed partition $\mathcal{I}'_*$ a refinement of another one $\mathcal{I}_*$ if each $\mathcal{I}_t$ is a union of several $\mathcal{I}'_u$. Given such partitions $\mathcal{I}_*, \mathcal{I}'_*$, let $(\mathbf{i}, \mathbf{a} = \mathbf{a}(M))$ (resp. $(\mathbf{i}', \mathbf{a}' = \mathbf{a}'(M))$) be the resulting monomial functions. It is easy to see that a flag in $\mathcal{F}_{\mathbf{i}',\mathbf{a}'}$ is a refinement of a flag in $\mathcal{F}_{\mathbf{i},\mathbf{a}}$. Thus, we have a canonical projection $\mathcal{F}_{\mathbf{i}',\mathbf{a}'} \to \mathcal{F}_{\mathbf{i},\mathbf{a}}$, an associated inclusion $P_{\mathbf{i}',\mathbf{a}'} \to P_{\mathbf{i},\mathbf{a}}$ and an inclusion $Y_{\mathbf{i}',\mathbf{a}'} \to Y_{\mathbf{i},\mathbf{a}}$. Using the realization of $X_{\mathbf{i},\mathbf{a}}$ as $G_d \times^{P_{\mathbf{i},\mathbf{a}}} Y_{\mathbf{i},\mathbf{a}}$, we see that we arrive at the following statement:

Lemma 2.4 *There exists a projection $X_{\mathbf{i}',\mathbf{a}'} \to X_{\mathbf{i},\mathbf{a}}$, which is compatible with the maps $\pi_{\mathbf{i}',\mathbf{a}'}$ and $\pi_{\mathbf{i},\mathbf{a}}$.*

Remark: The desingularization given by a directed partition $\mathcal{I}_*$ is thus 'optimal' if $\mathcal{I}_*$ is as coarse as possible. This leads to the notion of a regular partition in the sense of [Re1].

3 Construction of stratifications

In this section, we construct a decomposition of the fibres of the maps $\pi_{\mathbf{i},\mathbf{a}}$ and of the so-called orbital varieties $Y_{\mathbf{i},\mathbf{a}} \cap \mathcal{O}_N$ into disjoint, irreducible, smooth subvarieties. We will use the term stratification in this broad sense. In fact, this construction works for arbitrary monomials $(\mathbf{i}, \mathbf{a})$, not only for the ones constructed in the previous section. This special case will be studied in more detail at the end of the following section. So fix a monomial $(\mathbf{i}, \mathbf{a})$ and denote by $\mathcal{F}, P, Y, X, \pi$ the objects $\mathcal{F}_{\mathbf{i},\mathbf{a}}, P_{\mathbf{i},\mathbf{a}}, \ldots$. First we show that it suffices to stratify the orbital varieties $Y \cap \mathcal{O}_N$.

Lemma 3.1 *For all $N \in R_d$, we have G_d-equivariant isomorphisms:*

$$G_d \times^{\mathrm{Aut}_{kQ}(N)} \pi^{-1}(N) \simeq \pi^{-1}(\mathcal{O}_N) \simeq G_d \times^P (Y \cap \mathcal{O}_N).$$

Proof: The restriction $\pi|_{\pi^{-1}\mathcal{O}_N} : \pi^{-1}\mathcal{O}_N \to \mathcal{O}_N$ is a G_d-equivariant projection onto the homogeneous space $\mathcal{O}_N \simeq G_d/\mathrm{Aut}_{kQ}(N)$, yielding the first isomorphism. For the second one, we interpret the map π as

$$\pi : G_d \times^P Y \to R_d, \;\; \overline{(g, y)} \mapsto gyg^{-1}.$$

Thus, the subvariety $\pi^{-1}\mathcal{O}_N \subset R_d$ identifies with

$$\{\overline{(g, y)} \, : \, gyg^{-1} \in \mathcal{O}_N\} = \{\overline{(g, y)} \, : \, y \in \mathcal{O}_N\} = G_d \times^P (Y \cap \mathcal{O}_N).$$

□

Using these isomorphisms, we can translate geometric properties between the fibres of π and the orbital varieties. In particular, we can define our wanted stratification for $Y \cap \mathcal{O}_N$, which will simplify the proofs.

We define the stratification in terms of the representation theory of Q first:

Definition 3.2 *Choose an arbitrary sequence $(N = [N_0], \ldots, [N_\nu] = 0)$ of isomorphism classes of representations of Q and define*

$$\mathcal{S}_{[N_*]} = \mathcal{S}_{([N_0],\ldots,[N_\nu])} = \{L \in Y \, : \, L|_{F_0^k} \simeq N_k \text{ for all } k = 0 \ldots \nu\}.$$

This is well-defined since Y consists of those representations which are compatible with the flag F_0^*; in particular, each $L|_{F_0^k}$ for $L \in Y$ is again a representation of Q. Obviously, we have

$$Y = \bigcup_{N_*} \mathcal{S}_{[N_*]} \text{ and } Y \cap \mathcal{O}_N = \bigcup_{[N_*]:N_0 \simeq N} \mathcal{S}_{[N_*]}.$$

It is also clear that

$$\mathcal{S}_{[N_*]} = \emptyset \text{ unless } \underline{\dim}\, N_k = d^k := \sum_{l>k} a_l i_l \in \mathbf{N}I \text{ for all } k = 0 \ldots \nu.$$

Since there are only finitely many isoclasses of representations of a fixed dimension vector, we see that the above decomposition is finite.

Proposition 3.3 *The strata $\mathcal{S}_{[N_*]}$ are irreducible, smooth subvarieties of $Y \cap \mathcal{O}_N$.*

Proof: First, we will derive a geometric construction of the strata. Denote by Y_1 the subspace of R_d of representations which are compatible with F_0^1 only:

$$Y_1 = \{N \in R_d \, : \, N(F_0^1) \subset F_0^1\}.$$

The natural projection $p : Y_1 \to R_{d^1}$, which maps $N \in Y_1$ to the restriction $N|_{F_0^1}$, is a trivial vector bundle. It restricts to a projection $\overline{p} : Y_1 \cap \mathcal{O}_N \to R_{d^1}$. It is easy to see from the definitions that

$$\mathcal{S}_{[N_*]} = \overline{p}^{-1}(\mathcal{S}_{[N'_*]}),$$

where $[N'_*]$ denotes the truncated sequence $([N_1], \ldots, [N_\nu])$. Thus, we can construct the stratum $\mathcal{S}_{[X_*]}$ inductively, starting from $\mathcal{S}_{([X_\nu])} = R_{d^k} = R_0 = 0$.

We will now study the morphism $\overline{p} : Y_1 \cap \mathcal{O}_N \to R_{d^1}$ in more detail. The stratum $\mathcal{S}_{[N'_*]}$ is contained in $\mathcal{O}_{N_1}$, thus we only need to study the induced morphism $\overline{p} : \overline{p}^{-1}\mathcal{O}_{N_1} \to \mathcal{O}_{N_1}$. The group G_{d^1} acts on $\mathcal{O}_{N_1}$, and on $\overline{p}^{-1}\mathcal{O}_{N_1}$ as a subgroup of the parabolic in G_d leaving F_0^1 stable. Since $\overline{p}$ is obviously equivariant for these actions, we have

$$\overline{p}^{-1}(\mathcal{O}_{N_1}) \simeq G_{d^1} \times^{\mathrm{Aut}_{kQ}(N_1)} \overline{p}^{-1}(N_1).$$

Therefore, it suffices to study the fibre $\overline{p}^{-1}(N_1)$, which consists of all points y of $\mathcal{O}_N$ which are compatible with F_0^1 and restrict to N_1 on this subspace. The inverse image of $\overline{p}^{-1}(N_1)$ under the quotient map

$$q : G_d \to \mathcal{O}_N, \; q(g) = g^{-1}Ng$$

equals the set of all $g \in G_d$ such that $g^{-1}Ng$ is compatible with F_0^1 and restricts to N_1 on this subspace. An easy calculation shows that these conditions are equivalent to $Nf = fN_1$, where f denotes the induced homomorphism $f = g|_{F_0^1} : F_0^1 \to k^d$. On the other hand, we can consider the morphism

$$pr : G_d \to \mathrm{IHom}_k(F_0^1, k^d) = \{f \in \mathrm{Hom}_k(F_0^1, k^d) \, : \, f \text{ injective}\}$$

given by restriction of an automorphism of k^d to F_0^1, which is a flat morphism since it is open in the trivial vector bundle $\mathrm{End}_k(k^d) \to \mathrm{Hom}(F_0^1, k^d)$. The variety $\mathrm{IHom}_k(F_0^1, k^d)$ contains the subvariety of injective homomorphisms of Q-representations $\mathrm{IHom}_{kQ}(N_1, N)$, and its inverse image under pr is precisely $q^{-1}\overline{p}^{-1}(N_1)$, as calculated above. Thus, the varieties $\mathrm{IHom}_{kQ}(N_1, N)$ and $\overline{p}^{-1}(N_1)$ are related (via $q^{-1}\overline{p}^{-1}(N_1)$) by a quotient morphism on one side and a flat morphism on the other side. Since $\mathrm{IHom}_{kQ}(N_1, N)$ is an open subset of the vector space $\mathrm{Hom}_{kQ}(N_1, N)$, it is an irreducible smooth variety. We

conclude that the same holds for $\overline{p}^{-1}(N_1)$. By induction, we see that the proposition is proved.

$\square$

The above construction also allows us to compute the dimension of each stratum $\mathcal{S}_{[N_*]}$.

Proposition 3.4 *If the stratum $\mathcal{S}_{[N_*]}$ is non-empty, then it is of dimension*

$$\dim \mathcal{S}_{[N_*]} = \dim P_{\mathbf{i},\mathbf{a}} + \sum_{k=1}^{\nu} (\dim \mathrm{Hom}(N_k, N_{k-1}) - \dim \mathrm{End}(N_{k-1})).$$

Proof: The morphism q defined in the previous proof has relative dimension $\dim \mathrm{Aut}_{kQ}(N) = \dim \mathrm{End}(N)$, and the morphism pr has relative dimension

$$\sum_{i \in I} (d - d^1)_i d_i = a_1 d_{i_1} = \sum_{k=2}^{\nu} a_1 a_k.$$

Thus, we have

$$\begin{aligned} \dim \overline{p}^{-1}(N_1) &= \dim q^{-1}\overline{p}^{-1}(N_1) - \dim \mathrm{End}(N) \\ &= \underbrace{\dim \mathrm{IHom}_{kQ}(N_1, N)}_{=\dim \mathrm{Hom}(N_1,N)} - \dim \mathrm{End}(N) + \sum_{k=2}^{\nu} a_1 a_k. \end{aligned}$$

By induction, this yields the formula stated in the proposition.

$\square$

Concerning the structure of the closure of the strata, we prove, at the moment, only the following general criterion. A more detailed study will follow in the next section.

Proposition 3.5 *If $\mathcal{S}_{[L_*]}$ belongs to the closure $\overline{\mathcal{S}_{[N_*]}}$, then L_k belongs to the orbit closure $\overline{\mathcal{O}_{N_k}}$, and $\mathcal{S}_{[L_k],\ldots,[L_\nu]}$ belongs to the closure of $\mathcal{S}_{[N_k],\ldots,[N_\nu]}$ for all $k = 0 \ldots \nu$.*

Proof: Suppose $\mathcal{S}_{[L_*]} \subset \overline{\mathcal{S}_{[N_*]}}$. Considering the G_d-saturation of both sides, we get immediately that N_0 degenerates to L_0. Moreover, we have a chain of inclusions

$$\mathcal{S}_{[L'_*]} = \overline{p}(\mathcal{S}_{[L_*]}) \subset \overline{p}(\overline{\mathcal{S}_{[N_*]}}) \subset \overline{\overline{p}(\mathcal{S}_{[N_*]})} = \overline{\mathcal{S}_{[N'_*]}}.$$

In this chain, the equalities follow from the properties of the morphism $\overline{p}$, and the inclusions are obvious. By induction, the proposition follows.

$\square$

4 Geometry and combinatorics of the stratification

We continue to use the notations of the previous section. Using the relation between the fibres of π and the orbital varieties established before, we will now translate the results of the previous section to study the geometry of the singular fibres.

Recall from Lemma 3.1 the isomorphism

$$G_d \times^{\mathrm{Aut}_{kQ}(N)} \pi^{-1}(N) \simeq G_d \times^P (Y \cap \mathcal{O}_N).$$

By general properties of associated fibre bundles, we can make the following definition.

Definition 4.1 *Given a sequence* $[N_*] = ([N] = [N_0], \ldots, [N_\nu] = 0)$ *as before, define the stratum* $\mathcal{F}_{[N_*]}$ *in* $\pi^{-1}(N)$ *via the above isomorphism by*

$$G_d \times^{\mathrm{Aut}_{kQ}(N)} \mathcal{F}_{[N_*]} \simeq G_d \times^P \mathcal{S}_{[N_*]}.$$

The results of the previous section are immediately translated into the following:

Theorem 4.2

1. *The subsets* $\mathcal{F}_{[N_*]}$ *for various sequences* $[N_*]$ *define a stratification of* $\pi^{-1}(N)$ *into irreducible, smooth, locally closed subsets.*
2. *The stratum* $\mathcal{F}_{[N_*]}$ *is non-empty if and only if there exist short exact sequences* $0 \to N_k \to N_{k-1} \to E_{i_k}^{a_k} \to 0$ *for all* $k = 1 \ldots \nu$.
3. *If* $\mathcal{F}_{[N_*]}$ *is non-empty, then its dimension equals*

$$\dim \mathcal{F}_{[N_*]} = \sum_{k=1}^{\nu} (\dim \mathrm{Hom}(N_k, N_{k-1}) - \dim \mathrm{End}(N_k)).$$

4. *If the closure of a stratum* $\mathcal{F}_{[L_*]}$ *contains another one* $\mathcal{F}_{[N_*]}$*, then for all* $k = 1 \ldots \nu$*, we have* $L_k \leq N_k$ *and* $\overline{\mathcal{F}_{([L_k], \ldots, [L_\nu])}} \supset \mathcal{F}_{([N_k], \ldots, [N_\nu])}$.

From this, we can derive a formula for the dimension of the singular fibres:

Corollary 4.3 *For each* N*, we have*

$$\dim \pi^{-1}(N) = \max_{[N_*] : \mathcal{F}_{[N_*]} \neq \emptyset} \sum_{k=1}^{\nu} (\dim \mathrm{Hom}(N_k, N_{k-1}) - \dim \mathrm{End}(N_k)).$$

Another application of the stratification concerns the determination of the irreducible components of the singular fibres $\pi^{-1}(N)$.

Proposition 4.4

1. *If I is an irreducible component of $\pi^{-1}(N)$, then there exists a sequence $[N_*]$ such that $I = \overline{\mathcal{F}_{[N_*]}}$.*

2. *The subvariety $\overline{\mathcal{F}_{[N_*]}}$ is an irreducible component of $\pi^{-1}(N)$ provided the following holds: If $[L_*]$ is a sequence such that*

$$L_k \leq N_k \text{ and } \dim \mathcal{F}_{([L_k],\ldots,[L_\nu])} \geq \dim \mathcal{F}_{([N_k],\ldots,[N_\nu])}$$

for all $k = 1 \ldots \nu$, then already $L_k \simeq N_k$ for all $k = 1 \ldots \nu$.

In the case of Springer's resolution, the strata of the singular fibres are in bijection to Young tableaux of a fixed shape (compare [Sp]), i.e. to paths in the Young graph, which has Young diagrams as vertices, and arrows corresponding to additions of a single box. Analoguously, we will now construct a graph structure on isomorphism classes of representations of Q whose paths of a certain type parametrize the strata in our stratification.

Definition 4.5 *Let $\Gamma(Q)$ be the $I \times \mathbf{N}$-coloured graph with vertices the isomorphism classes of representations of Q, and with an arrow of colour $(i,n) \in I \times \mathbf{N}$ from N to X if there exists an exact sequence $0 \to N \to X \to E_i^n \to 0$. A path of colour $(\mathbf{i},\mathbf{a})$ in $\Gamma(Q)$ is a sequence $[N_*]$ such that there exist arrows $[N_{k-1}] \to [N_k]$ of colour (i_k, a_k) for all $k = 1 \ldots \nu$.*

The second part of Theorem 4.2 now reads as follows:

Lemma 4.6 *The non-empty strata in $\pi_{\mathbf{i},\mathbf{a}}^{-1}(N)$ are parametrized by the paths of colour $(\mathbf{i},\mathbf{a})$ from $[0]$ to $[N]$ in $\Gamma(Q)$.*

For a certain class of quivers, which we call special, the strata can be parametrized in a completely combinatorial way, using results from [Re3]. Since this class of quivers was described there only in representation-theoretic terms, we first give a purely root-theoretic description.

Definition 4.7 *A quiver Q is called special if for all indecomposables X_α and all simples E_i, we have $\dim \mathrm{Hom}(X_\alpha, E_i) \leq 1$.*

Call a vertex $i \in I$ thick if there exists a root $\alpha \in R^+$ such that the simple root $i \in R^+$ appears with multiplicity at least 2 in α.

Proposition 4.8 *Q is special if and only if no thick vertex is a source of Q.*

Proof: First, we prove the following statement:
Q is special if and only if $\langle \alpha, i \rangle \leq 1$ for all $\alpha \in R^+$ and all $i \in I$.
So suppose Q to be special, and let $\alpha \in R^+$ and $i \in I$. Then

$$\langle \alpha, i \rangle = \dim \mathrm{Hom}(X_\alpha, E_i) - \dim \mathrm{Ext}^1(X_\alpha, E_i) \leq \dim \mathrm{Hom}(X_\alpha, E_i) \leq 1.$$

Conversely, suppose X_α and E_i as above are given. If $\mathrm{Hom}(X_\alpha, E_i) = 0$, there is nothing to prove. Otherwise, we already have $\mathrm{Ext}^1(X_\alpha, E_i) = 0$ by the directedness of modkQ, so

$$1 \geq \langle \alpha, i \rangle = dim\mathrm{Hom}(X_\alpha, E_i) - \dim \mathrm{Ext}^1(X\alpha, E_i) = \dim \mathrm{Hom}(X_\alpha, E_i).$$

Now we come to the proof of the proposition. First we show that the above condition is neccessary. So let i be a thick vertex of Q, and let $\alpha = \sum_{i \in I} \alpha_i i \in R^+$ be a root such that $\alpha_i \geq 2$. By the previous lemma, we have

$$1 \geq \langle \alpha, i \rangle = \alpha_i - \sum_{j \to i} \alpha_j \geq 2 - \sum_{j \to i} \alpha_j,$$

so that i cannot be a source in Q.

To prove the converse, we have to proceed by a case-by-case analysis:
If Q is of type A_n, then α_i equals 0 or 1 for all $\alpha \in R^+$ and $i \in I$. This means that no vertex is thick, and that Q is always special. If Q is of type D_n, we have the following property: If α_i=2, then $\alpha_j = 1$ for all j connected to i. Thus, we are done by the above inequality, assuming that no thick i is a source. If Q is of type E_6 or E_7, then a direct inspection of the set of positive roots shows the converse. If Q is of type E_8, then all vertices are thick (look at the highest root $\alpha = (2, 3, 4, 6, 5, 4, 3, 2)$), so there is no orientation Q for which no thick vertex is a source, and thus there is no special orientation.

□

Assume now that Q is special. For each vertex $i \in I$, define a partially ordered set $\mathcal{P}_i \subset R^+$ as the set of all positive roots α such that $\langle \alpha, i \rangle = 1$, partially ordered by $\alpha \leq \beta$ if $\langle \alpha, \beta \rangle \geq 1$ (compare [Re3], Proposition 4.3). Let $\mathcal{S}_i$ be the set of antichains in $\mathcal{P}_i$; it is naturally in bijection to the poset of order ideals in $\mathcal{P}_i$, from which it inherits a partial ordering. Let $\tau \in W$ be the Weyl group element defined by $X_{\tau\alpha} = \tau X_\alpha$. Given an antichain $A \in \mathcal{S}_i$, we denote by $l(A)$ the set of all $\alpha \in \mathcal{P}_i$ which are minimal with the property $\{\alpha\} \not\leq A$. Translating Proposition 4.5 of [Re3] into a root-theoretic language, we get:

Proposition 4.9 *For any representation $M \in$ modkQ, the possible middle terms X of exact sequences $0 \to M \to X \to E_i \to 0$ are parametrized by antichains $A \in \mathcal{S}_i$ such that $X_{\tau\alpha}$ is a direct summand of M for each $\alpha \in l(A)$. The corresponding middle term is given by*

$$X = B \oplus \bigoplus_{\alpha \in A} X_\alpha, \text{ where } M = B \oplus \bigoplus_{\alpha \in l(A)} X_{\tau\alpha}.$$

Repeated application of this proposition yields all possible middle terms of exact sequences $0 \to M \to X \to E_i^n \to 0$. Combining this with Lemma 4.6, we get:

Corollary 4.10 *If Q is special, there exists a purely root-theoretic parametrization of the strata in each $\pi_{\mathbf{i},\mathbf{a}}(N)$.*

Remark: Using the proof of Proposition 5.2 of [Re3], one could also give a combinatorial formula for the dimension of the singular fibres.

Example: We consider again the desingularization from the first example of section 2, and use the notations defined there. Let N be a point in $\overline{\mathcal{O}_M}$, and assume N is given by multiplicities $(n_{ij})_{i\leq j}$. By the above proposition, all exact sequences with simple right end term are of the form $0 \to B \oplus E_{i+1,j} \to B \oplus E_{ij} \to E_i$ for some $j \geq i$. Thus, exact sequences with right end term E_i^n are of the form

$$0 \to B \oplus \bigoplus_{j\geq i} E_{i+1,j}^{p_j} \to B \oplus \bigoplus_{j\geq i} E_{ij}^{p_j} \to E_i^n \to 0.$$

It follows that the sequences $[N_*]$ defining non-empty strata $\mathcal{F}_{[N_*]}$ in $\pi_M^{-1}(N)$ are parametrized by tuples (p_k^{ij}) for $i \leq j \leq k$ such that

$$\sum_k p_k^{ij} = m_{ij} + \ldots + m_{in}, \quad \sum_k (p_j^{ik} - p_j^{k,i-1}) = n_{ij} \text{ for all } i \leq j,$$

$$\sum_{l\leq i} (p_k^{lj} - p_k^{l,j+1}) \leq 0 \text{ for all } i \leq j < k.$$

Now we turn to the special case where $(\mathbf{i}, \mathbf{a})$ is a monomial function constructed from some directed partition. In this case, the stratification of the singular fibres of π_M can be viewed as a graded analogue of Spaltenstein's stratification of [Sp]. However, in our situation, the geometry of the singular fibres is considerably more complicated.

Remarks:

1. The fibres are far from being equidimensional, as the dimension formula from Theorem 4.2 already shows in examples for type A_3.

2. The irreducible components are not in bijection to the strata. In fact, one frequently encounters inclusions of closures of strata. At the moment, the best result for identifying irreducible components is Proposition 4.4; a representation-theoretic criterion for inclusions of closures of strata is missing. Nevertheless, in examples of type A_3 Proposition 4.4 detects roughly half the number of irreducible components.

3. The maps π_M are not semi-small (in the sense of intersection homology theory) in general, as can be seen in examples using Corollary 4.3. In fact, the results of the following section relate this property to monomiality of certain elements in Lusztig's canonical basis, which is discussed in detail in [Lu4], [Re4].

4. It is not clear whether the singular fibres are always connected. This follows immediately from Zariski's Main Theorem provided the orbit closures

$\overline{\mathcal{O}_M}$ are normal varieties. So far, this is only known for quivers of type A by [BZ], where it is reduced to the case of an equioriented quiver of type A, which was treated in [ADK].

In fact, the connectedness of the singular fibres is directly related to the question of normality (or at least unibranchness, which, together with a Frobenius splitting, implies normality) of the orbit closures, as the following lemma shows:

Lemma 4.11 *Let $f : X \to Y$ be a desingularization map between algebraic varieties X, Y. Then all $f^{-1}(y)$ are connected if and only if Y is unibranch, i.e. if the normalization morphism $g : \widetilde{Y} \to Y$ is bijective.*

We omit the proof. It follows easily from the universal property of the normalization, standard properties of finite morphism and Zariski's Main Theorem.

5 Relation to quantized enveloping algebras

In this section, we assume k to be a finite field with v^2 elements. Given a finite G-set X, i.e. a finite set on which a group G acts, we denote by $\mathbf{Q}_G[X]$ the set of G-invariant functions from X to the rationals. We define the Hall algebra of the quiver Q ([Ri2]) in terms of a convolution product. Denote by $H_v(Q)$ the direct sum

$$H_v(Q) = \bigoplus_{d \in \mathbf{N}I} \mathbf{Q}_{G_d}[R_d],$$

where each R_d is viewed as a finite G_d-set over the finite field k. Define a multiplication on $H_v(Q)$ by

$$(f * g)(X) = v^{\langle d,e \rangle} \sum_{U \subset X} f(X/U) \cdot G(U),$$

where $f \in \mathbf{Q}_{G_d}[R_d]$, $g \in \mathbf{Q}_{G_e}[R_e]$, and the sum runs over all subrepresentations of X, which is viewed as an object of $\bmod kQ$. This product is well defined since the functions f and g only depend on the isomorphism class of a representation; it endows $H_v(Q)$ with the structure of an associative, $\mathbf{N}I$-graded $\mathbf{Q}$-algebra. Denote by E_i the function with value $1 \in \mathbf{Q}$ on the one-point set R_i. The main result of [Ri2] can be stated in this language as:
The map $\eta : E_i \mapsto E_i$ extends to an isomorphism of $\mathbf{N}I$-graded $\mathbf{Q}$-algebras

$$\eta : H_v(Q) \xrightarrow{\sim} \mathcal{U}_v(\mathfrak{n}^+),$$

where $\mathcal{U}_v(\mathfrak{n}^+)$ denotes the positive part of the quantized enveloping algebra to the Dynkin diagram underlying Q, which is a $\mathbf{Q}$-algebra with generators E_i for $i \in I$, subjected to the quantized Serre-relations at v.

Analoguosly to the previous sections, we fix a directed partition $(\mathcal{I}_*)$ of R^+ and

denote by $(\mathbf{i}, \mathbf{a})$ the corresponding monomial function. Given a representation M of dimension vector d, we thus have a monomial $(\mathbf{i}, \mathbf{a} = (a_1 \dots a_\nu))$ and the corresponding desingularization $\pi_M : X_M \to \overline{\mathcal{O}_M}$.

Definition 5.1 *Define a monomial in $H_v(Q)$ by*

$$E^{(M)} = E_{i_1}^{(a_1)} * E_{i_2}^{(a_2)} * \dots * E_{i_\nu}^{(a_\nu)} \in \mathbf{Q}_{G_d}[R_d],$$

where $E_i^{(n)}$ denotes the divided power $([n]!)^{-1} E_i^{*n} \in \mathbf{Q}_{G_{ni}}[R_{ni}]$.

Lemma 5.2 *For all representation M, N, we have*

$$E^{(M)}(N) = v^{\dim \mathrm{End}(M,M) - \dim M} |\pi_M^{-1}(N)|,$$

where $|\cdot|$ denotes the cardinality of a finite set.

Proof: The value of the function $E_i^{(n)}$ on the unique point 0 of R_{ni} is easily computed from the definitions as

$$E_i^{(n)}(0) = ([n]!]^{-1} E_i^{*n}(0) = ([n]!)^{-1} v^{n(n-1)/2} |\mathcal{F}_n(k)| = v^{n(n-1)},$$

where $\mathcal{F}_n$ denotes the set of complete flags in the vector space k^n. Using this, we can compute the value of a monomial in the $E_i^{(n)}$ as $E_{i_1}^{(a_1)} * E_{i_2}^{(a_2)} * \dots * E_{i_\nu}^{(a_\nu)}(N) =$

$$\begin{aligned}
&= v^{\sum_{k<l} \langle a_k i_k, a_l i_l \rangle} \sum_{N = N_0 \supset \dots \supset N_\nu = 0} E_{i_1}^{(a_1)}(N_0/N_1) \cdot \ldots \cdot E_{i_\nu}^{(a_\nu)}(N_{\nu-1}/N_\nu) \\
&= v^C |\{N_* = (N = N_0 \supset \dots \supset N_\nu = 0) : \underline{\dim}\, N_{k-1}/N_k = a_k i_k \text{ for all k}\}| \\
&= v^C |\pi_M^{-1}(N)|
\end{aligned}$$

where $C = \sum_{k \leq l} \langle a_k i_k, a_l i_l \rangle - \sum_k a_k$. Thus, it remains to identify the exponents of v. We have

$$\dim \mathrm{End}(M, M) = \dim G_d - \dim \mathcal{O}_M = \dim G_d - \dim X_M;$$

using the dimension formulae of section 2 and the definition of the bilinear form $\langle , \rangle$, this reduces to an easy calculation.

□

The set of elements $B = \{E_M : [M] \in [\mathrm{mod} kQ]\}$ of $H_v(Q)$ defined by

$$E_M(N) = \begin{cases} v^{\dim \mathrm{End}(M,M) - \dim M} & , \quad M \simeq N, \\ 0 & , \quad \text{otherwise} \end{cases}$$

is obviously a basis for $H_v(Q)$. It is proved in [Lu1] that the corresponding basis for $\mathcal{U}_v(\mathfrak{n}^+)$ is of PBW type. The above lemma can now be rewritten as

Proposition 5.3 *For each representation M, we have*

$$E^{(M)} = \sum_{[N]} |\pi^{-1}(N)| E_N.$$

In particular, the set of elements $\mathcal{M} = \{E^{(M)} : [M] \in [\mathrm{mod} kQ]\}$ is a monomial basis for $H_v(Q)$, which has upper unitriangular base change (with respect to the degeneration ordering) to the PBW basis B.

Proof: The base change coefficients are precisely calculated by the preceding lemma. We have $|\pi_M^{-1}(M)| = 1$, and $|\pi_M^{-1}(N)| \neq 0$ only if $M \leq N$ since π_M is a desingularization of $\overline{\mathcal{O}_M}$. This implies the unitriangularity property; in particular, the $E^{(M)}$ already form a basis for $H_v(Q)$.

□

Remark: It can easily be seen from the definitions that the monomial bases thus constructed are exactly the same as in [Re1].

Now we consider the relation of the monomial bases to Lusztig's canonical basis. For each isoclass $[M]$ in $\mathrm{mod} kQ$, we introduce an element $\mathcal{E}_M$ in $H_v(Q)$ by

$$\mathcal{E}_M(N) = v^{\dim \mathrm{End}(M) - \dim M} \sum_{i \in \mathbf{Z}} \dim \mathcal{H}^i_N \mathcal{IC}(\mathcal{O}_M) v^i,$$

where $\mathcal{H}^i_N \mathcal{IC}(\mathcal{O}_M)$ denotes the stalk at the point N of the i-th cohomology sheaf of the intersection cohomology complex corresponding to the orbit of M as a variety over $k = \mathbf{C}$. By general properties of perverse sheaves, these elements have upper unitriangular base change (with respect to the degeneration ordering) to the PBW basis, thus $\mathcal{B} = \{\mathcal{E}_M : [M] \in [\mathrm{mod} kQ]\}$ is a basis for $H_v(Q)$.

Proposition 5.4 *The base change from $\mathcal{M}$ to $\mathcal{B}$ is upper unitriangular with entries consisting of Laurent polynomials with non-negative integer coefficients.*

Proof: The proof uses Lusztig's realization of $\mathcal{U}_v(\mathfrak{n}^+) \simeq H_v(Q)$ in terms of convolution of perverse sheaves ([Lu3], 13.2.11.). Combining Lemma 5.2 with ([Lu3], 9.1.3.) and ([Lu3], 13.1.12.(b)), one can interprete the base change coefficients from $\mathcal{M}$ to $\mathcal{B}$ as multiplicities of simple perverse sheaves in the decomposition of the derived direct image $(\pi_M)_* 1$ of the constant sheaf on X_M (up to some shifts). From this, both the positivity property and the unitriangularity property follow immediately (since π_M is a desingularization).

□

6 Convolution algebras

In this section, we assume $k = \mathbf{C}$. Fix $\mathcal{I}_*$, $\mathbf{i}$ and $\mathbf{a}$ as in the previous sections. Given a representation M, we have the corresponding desingularization π_M :

$X_M \to \overline{\mathcal{O}_M}$. We define

$$Z_M := X_M \times_{\overline{\mathcal{O}_M}} X_M.$$

This is an analogue of the triple variety in the context of Springer's desingularization: Using the description of X_M of section 2, it is easy to see that

$$Z_M \simeq \{(N, F_1^*, F_2^*) \in R_d \times \mathcal{F}_{\mathbf{i},\mathbf{a}} \times \mathcal{F}_{\mathbf{i},\mathbf{a}} \,:\, NF_i^* \subset F_i^* \text{ for } i = 1, 2\}.$$

We recall some results on convolution algebras from [CG]. The total Borel-Moore homology

$$A_M = H_*^{\mathrm{BM}}(Z_M, \mathbf{C})$$

carries a natural associative algebra structure via convolution. It acts via convolution on the total Borel-Moore homology $V_N = H_*^{\mathrm{BM}}(\pi_M^{-1}(N), \mathbf{C})$ of the fibres of the desingularization π_M. The simple A_M-modules are parametrized by a subset of the set of isoclasses $[N] \in \mathrm{mod}kQ$ such that N is a degeneration of M. In fact, each corresponding simple module L_N can be realized as a quotient of V_N (this follows from ([CG], chapter 8), since in our situation, all orbits $\mathcal{O}_N$ are simply connected (the stabilizer $\mathrm{Aut}_{kQ}(M)$ is conneced). The result of [CG] which we need here mainly is Theorem 8.6.23, stating that the decomposition numbers of the representations V_N are given by Poincare polynomials in local intersection cohomology:

$$[V_{N_1} : L_{N_2}] = \sum_{i \in \mathbf{Z}} \dim \mathcal{H}^i_{N_1} \mathcal{IC}(\mathcal{O}_{N_2}),$$

where $[V_{N_1} : L_{N_2}]$ denotes the Jordan-Hölder multiplicity of the simple A_M-module L_{N_2} in V_{N_1}.

All objects considered in the previous section have 'generic' versions: In the $\mathbf{Q}(v)$-algebra $\mathcal{U}^+$ given by generators E_i for $i \in I$ subjected to the quantized Serre relations, one has elements $E_M, \mathcal{E}_M, E^{(M)}$ which specialize to the corresponding basis elements for each v such that v^2 is the cardinality of a finite field ([Lu3]). In particular, the enveloping algebra $\mathcal{U}(\mathfrak{n}^+)$ is provided with both canonical and PBW bases.

This allows us to prove the following 'Kazhdan-Lusztig type' statement:

Theorem 6.1 *Writing $\mathcal{E}_M = \sum_{[N]} \zeta_N^M E_N$ in the enveloping algebra $\mathcal{U}(\mathfrak{n}^+)$, we have*

$$\zeta_N^M = [V_N : L_M].$$

Proof: Since π_M is a desingularization, the fibre over M is a single point, thus $V_M = L_M$ is already a simple (1-dimensional) A_M-module. The result of [CG] cited above identifies the decomposition number $[V_N : L_M]$ with $\sum_{i \in \mathbf{Z}} \dim \mathcal{H}^i_N \mathcal{IC}(\mathcal{O}_M)$. But by the definition of the canonical basis given in the previous section, this is just the $v = 1$-specialization of the base change coefficient to the PBW basis.

□

The immediate problem of an algebraic description of the algebras A_M arises. At the moment, no general descriptions are known. The analysis of their structure is planned for future publications. At this point, we discuss one particular example.

Example: Let Q be the quiver $i \to j$ of type A_2, and let M be the representation $E_j^{n-1} \oplus E_{ij} \oplus E_i^{m-1}$, so that $\overline{\mathcal{O}_M}$ is the set of complex $m \times n$-matrices of rank at most 1. Then A_M has generators x, y, subjected to the relations

$$x^n = 0,\ yx^k y = \delta_{k,n-m-1} \cdot y \text{ for all } k \geq 0,\ x^m = \sum_{p+q=n-1} x^p y x^q.$$

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